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CONVEX POLYHEDRA WITH REGULAR FACES

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1. Introduction. An interesting set of geometric figures is composed of the convex polyhedra in Euclidean 3-space whose faces are regular polygons (not necessarily all of the same kind). A polyhedron with regular faces is *uniform* if it has symmetry operations taking a given vertex into each of the other vertices in turn (**5**, p. 402). If in addition all the faces are alike, the polyhedron is *regular*.

That there are just five convex regular polyhedra—the so-called Platonic solids—was proved by Euclid in the thirteenth book of the *Elements* (**10**, pp. 467–509). Archimedes is supposed to have described thirteen other uniform, "semi-regular" polyhedra, but his work on the subject has been lost. Kepler (**12**, pp. 114–127) showed that the convex uniform polyhedra consist of the Platonic and Archimedean solids together with two infinite families—the regular prisms and antiprisms. It was Kepler also who gave the Archimedean polyhedra their generally accepted names.

It is fairly easy to show that there are only a finite number of non-uniform regular-faced polyhedra (**11**; **13**), but it is no simple matter to establish the exact number. However, it appears that there are just ninety-two such solids. Some special cases were discussed by Freudenthal and van der Waerden (**8**), and a more general treatment was attempted by Zalgaller (**13**). Subsequently, Zalgaller *et al.* (**14**) determined all the regular-faced polyhedra having one or more trivalent vertices and all those having only pentavalent vertices. Grünbaum and Johnson (**9**) proved that the only kinds of faces that a regular-faced solid, other than a prism or an antiprism, may have are triangles, squares, pentagons, hexagons, octagons, and decagons.

A regular n-gon is conveniently denoted by the *Schläfli symbol* $\{n\}$. Thus $\{3\}$ is an equilateral triangle, $\{4\}$ a square, $\{5\}$ a regular pentagon, etc. An edge of a regular-faced polyhedron common to an $\{m\}$ and an $\{n\}$ will be said to be of type $\langle m \cdot n\rangle$. The sum of the face angles at a vertex of a convex polyhedron must be less than 360°. If the faces are regular, it follows that no more than five can meet at any vertex; in other words, each vertex of a convex polyhedron with regular faces must be trivalent, tetravalent, or pentavalent. Various combinations of faces give rise to many different types of vertices. For example:

$(4\cdot6\cdot8)$—a square, a hexagon, and an octagon;
$(3\cdot4\cdot3\cdot6)$—a triangle, a square, a triangle, and a hexagon;
$(3^2\cdot4\cdot6)$—two triangles, a square, and a hexagon;
$(3^4\cdot5)$—four triangles and a pentagon.

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The symbol $(4\cdot 6\cdot 8)$ could also be written $(6\cdot 8\cdot 4)$, $(4\cdot 8\cdot 6)$, etc. Note, however, that $(3\cdot 4\cdot 3\cdot 6)$ and $(3^2\cdot 4\cdot 6)$ are not equivalent, since the two triangles are separated in the one case but adjacent in the other.

2. Uniform polyhedra. Since a uniform polyhedron is completely characterized by the faces that surround one of its vertices, the vertex-type symbol, without parentheses, may be used as a symbol for the polyhedron (**2**, pp. 107, 130ff.; **3**, p. 394; **7**, pp. 56–57). The triangular prism, for example, is $3\cdot 4^2$. However, an extension of the Schläfli symbol devised by Coxeter (**3**, pp. 394–395; **5**, pp. 403–404) reveals more clearly the relationships between polyhedra. In this notation,

$\{m, n\}$ is the regular polyhedron whose faces are $\{m\}$'s, n surrounding each vertex, i.e., the polyhedron whose vertices are of type (m^n):

$\{3, 3\}$ is the *tetrahedron*;
$\{3, 4\}$ is the *octahedron*;
$\{4, 3\}$ is the *cube*;
$\{3, 5\}$ is the *icosahedron*;
$\{5, 3\}$ is the *dodecahedron*.

If we let

$$N_0 = \frac{4m}{4-(m-2)(n-2)}, \quad N_1 = \frac{2mn}{4-(m-2)(n-2)},$$

$$N_2 = \frac{4n}{4-(m-2)(n-2)}$$

(**4**, p. 13), then $\{m, n\}$ has N_0 vertices, N_1 edges, and N_2 faces.

$\begin{Bmatrix} m \\ n \end{Bmatrix}$ is the quasi-regular polyhedron with N_2 faces $\{m\}$, N_0 faces $\{n\}$, $2N_1$ edges $\langle m\cdot n\rangle$, and N_1 vertices $(m\cdot n\cdot m\cdot n)$:

$\begin{Bmatrix} 3 \\ 3 \end{Bmatrix} = \{3, 4\}$;

$\begin{Bmatrix} 3 \\ 4 \end{Bmatrix}$ is the *cuboctahedron*;

$\begin{Bmatrix} 3 \\ 5 \end{Bmatrix}$ is the *icosidodecahedron*.

$t\{m, n\}$ has N_0 faces $\{n\}$, N_2 faces $\{2m\}$, and $2N_1$ vertices $(n\cdot 2m\cdot 2m)$:

$t\{3, 3\}$ is the *truncated tetrahedron*;
$t\{3, 4\}$ is the *truncated octahedron*;
$t\{4, 3\}$ is the *truncated cube*;
$t\{3, 5\}$ is the *truncated icosahedron*;
$t\{5, 3\}$ is the *truncated dodecahedron*.

$\mathrm{r}\begin{Bmatrix} m \\ n \end{Bmatrix}$ has N_1 square faces, N_2 faces $\{m\}$, N_0 faces $\{n\}$, and $2N_1$ vertices $(m \cdot 4 \cdot n \cdot 4)$:

$$\mathrm{r}\begin{Bmatrix} 3 \\ 3 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 4 \end{Bmatrix};$$

$\mathrm{r}\begin{Bmatrix} 3 \\ 4 \end{Bmatrix}$ is the *rhombicuboctahedron;*

$\mathrm{r}\begin{Bmatrix} 3 \\ 5 \end{Bmatrix}$ is the *rhombicosidodecahedron.*

$\mathrm{t}\begin{Bmatrix} m \\ n \end{Bmatrix}$ has N_1 square faces, N_2 faces $\{2m\}$, N_0 faces $\{2n\}$, and $4N_1$ vertices $(4 \cdot 2m \cdot 2n)$:

$$\mathrm{t}\begin{Bmatrix} 3 \\ 3 \end{Bmatrix} = \mathrm{t}\{3, 4\};$$

$\mathrm{t}\begin{Bmatrix} 3 \\ 4 \end{Bmatrix}$ is the *truncated cuboctahedron;*

$\mathrm{t}\begin{Bmatrix} 3 \\ 5 \end{Bmatrix}$ is the *truncated icosidodecahedron.*

$\mathrm{s}\begin{Bmatrix} m \\ n \end{Bmatrix}$ has $2N_1$ triangular faces, N_2 faces $\{m\}$, N_0 faces $\{n\}$, and $2N_1$ vertices $(3^2 \cdot m \cdot 3 \cdot n)$:

$$\mathrm{s}\begin{Bmatrix} 3 \\ 3 \end{Bmatrix} = \{3, 5\};$$

$\mathrm{s}\begin{Bmatrix} 3 \\ 4 \end{Bmatrix}$ is the *snub cuboctahedron;*

$\mathrm{s}\begin{Bmatrix} 3 \\ 5 \end{Bmatrix}$ is the *snub icosidodecahedron.*

$\{\ \} \times \{n\}$ has n square faces separating two $\{n\}$'s, and $2n$ vertices $(4^2 \cdot n)$:

$\{\ \} \times \{4\} = \{4, 3\};$
$\{\ \} \times \{n\}$ $(n = 3, 5, 6, \ldots)$ is the *n-gonal prism.*

$\mathrm{h}\{\ \}\mathrm{s}\{n\}$ has $2n$ triangular faces separating two $\{n\}$'s, and $2n$ vertices $(3^3 \cdot n)$:

$\mathrm{h}\{\ \}\mathrm{s}\{3\} = \{3, 4\};$
$\mathrm{h}\{\ \}\mathrm{s}\{n\}$ $(n = 4, 5, 6, \ldots)$ is the *n-gonal antiprism.*

The prefix "rhomb(i)-" in the names for $\mathrm{r}\begin{Bmatrix} 3 \\ 4 \end{Bmatrix}$ and $\mathrm{r}\begin{Bmatrix} 3 \\ 5 \end{Bmatrix}$ derives from the fact that the former has 12 square faces whose planes bound a *rhombic dodecahedron,* the solid dual to $\begin{Bmatrix} 3 \\ 4 \end{Bmatrix}$, while the latter has 30 squares lying in the face-planes of a *rhombic triacontahedron,* the dual of $\begin{Bmatrix} 3 \\ 5 \end{Bmatrix}$.

Some persons object to the names "truncated cuboctahedron" and "trun-

cated icosidodecahedron" on the ground that actual truncations of $\begin{Bmatrix}3\\4\end{Bmatrix}$ or $\begin{Bmatrix}3\\5\end{Bmatrix}$ would have rectangular faces instead of squares. For this reason $4\cdot6\cdot8$ is sometimes called the "great rhombicuboctahedron," with $3\cdot4^3$ then being known as the "small rhombicuboctahedron," and similarly for $4\cdot6\cdot10$ and $3\cdot4\cdot5\cdot4$ (**2**, p. 138; **7**, p. 94). But this nomenclature is subject to the more serious objection that the words "great" and "small" have an entirely different connotation in connection with star polyhedra, as in the names of the Kepler–Poinsot solids (**2**, pp. 143–145; **5**, p. 410; **7**, pp. 83–93).

It should also be pointed out that $s\begin{Bmatrix}3\\4\end{Bmatrix}$ and $s\begin{Bmatrix}3\\5\end{Bmatrix}$ are more commonly known as the "snub cube" and the "snub dodecahedron," respectively, the names given them by Kepler. But it is clear that they are related just as closely to the octahedron and the icosahedron, and I have renamed them accordingly (cf. **2**, p. 138, or **3**, p. 395).

It is sometimes useful to consider the prisms and antiprisms as being derived from the fictitious polyhedra $\{2, n\}$ and $\begin{Bmatrix}2\\n\end{Bmatrix}$. Thus, for $n \geqslant 3$,

$$t\{2, n\} = r\begin{Bmatrix}2\\n\end{Bmatrix} = \{\} \times \{n\}, \qquad t\begin{Bmatrix}2\\n\end{Bmatrix} = \{\} \times \{2n\}, \qquad s\begin{Bmatrix}2\\n\end{Bmatrix} = h\{\}s\{n\},$$

where, in the case of $r\begin{Bmatrix}2\\n\end{Bmatrix}$ and $s\begin{Bmatrix}2\\n\end{Bmatrix}$, digonal "faces" are to be disregarded (cf. **5**, p. 403). Also,

$$t\begin{Bmatrix}2\\2\end{Bmatrix} = \{4, 3\} \quad \text{and} \quad s\begin{Bmatrix}2\\2\end{Bmatrix} = \{3, 3\}.$$

All the vertices of a uniform polyhedron are necessarily of the same type. However, the fact that a regular-faced solid has only one type of vertex does not guarantee that the solid is uniform, as is shown by the existence of a non-uniform polyhedron which, like the rhombicuboctahedron, has vertices all of type $(3\cdot4^3)$. This solid, depicted in Fig. 1, was discovered by J. C. P.

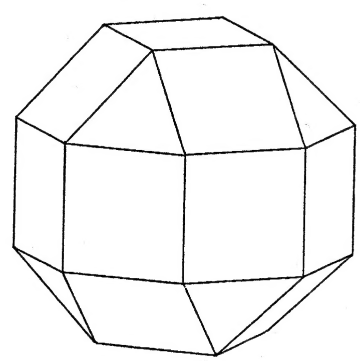


FIGURE 1

Miller sometime before 1930 (**2**, p. 137). More recently, Aškinuze (**1**) claimed that it should be counted as a fourteenth Archimedean polyhedron.

3. Cut-and-paste polyhedra. If a uniform polyhedron has a set of non-adjacent edges that form a regular polygon, then it is separated by the plane of this polygon into two pieces, each of which is a convex polyhedron with regular faces. This can be done with the octahedron and the icosahedron, the cuboctahedron and the icosidodecahedron, the rhombicuboctahedron and the rhombicosidodecahedron. In this manner or by using uniform polyhedra and pieces of uniform polyhedra as building blocks, eighty-three non-uniform regular-faced solids can be constructed.

An *n-gonal pyramid* Y_n ($n = 3, 4, 5$) has n triangular faces and one $\{n\}$, n vertices $(3^2 \cdot n)$ and one (3^n). A triangular pyramid is, of course, a tetrahedron. A square pyramid is half of $\{3, 4\}$, and a pentagonal pyramid is part of $\{3, 5\}$.

An *n-gonal cupola* Q_n ($n = 3, 4, 5$), obtainable as a fraction of $r\begin{Bmatrix}3\\n\end{Bmatrix}$, is a polyhedron having n triangular and n square faces separating an $\{n\}$ and a $\{2n\}$, with $2n$ vertices $(3 \cdot 4 \cdot 2n)$ and n vertices $(3 \cdot 4 \cdot n \cdot 4)$.

A *pentagonal rotunda* R_5, half of $\begin{Bmatrix}3\\5\end{Bmatrix}$, has 10 triangular and 5 pentagonal faces separating a $\{5\}$ and a $\{10\}$, 10 vertices $(3 \cdot 5 \cdot 10)$ and 10 vertices $(3 \cdot 5 \cdot 3 \cdot 5)$.

A pyramid, cupola, or rotunda is *elongated* if it is adjoined to an appropriate prism (a pentagonal pyramid to a pentagonal prism, a pentagonal cupola or rotunda to a decagonal prism, etc.) or *gyroelongated* if it is adjoined to an antiprism.

Two pyramids can be put together to form a *dipyramid*; two cupolas, a *bicupola*; a cupola and a rotunda, a *cupolarotunda*; and two rotundas, a *birotunda*. In the last three cases, the prefix *ortho-* is used to indicate that one of the two bases is the orthogonal projection of the other (as in a prism); the prefix *gyro-* indicates that one base is turned relative to the other (as in an antiprism). One of these polyhedra is elongated or gyroelongated when the two parts are separated by a prism or an antiprism.

Two triangular prisms can be joined to form a *gyrobifastigium*.

Certain uniform polyhedra can be *augmented* by adjoining other solids to them: square pyramids may be added to an n-gonal prism ($n = 3, 5, 6$), pentagonal pyramids to $\{5, 3\}$, and n-gonal cupolas to $t\{n, 3\}$ ($n = 3, 4, 5$). Other uniform polyhedra can be *diminished* by removing pieces of them—pentagonal pyramids from $\{3, 5\}$, pentagonal cupolas from $r\begin{Bmatrix}3\\5\end{Bmatrix}$. And pentagonal cupolas in $r\begin{Bmatrix}3\\5\end{Bmatrix}$ can be rotated 36° to produce a *gyrate* solid. Prefixes *bi-* and *tri-* are used where more than one piece is added, subtracted, or twisted. Where there are two different ways of adjoining, removing, or

turning a pair of pieces, these are distinguished by the further prefixes *para-* if the pieces are opposite each other and *meta-* if they are not.

To facilitate the description of regular-faced solids obtained from uniform polyhedra, certain of the latter are given abbreviated symbols:

$$\begin{array}{lll} S_2 = Y_3 = \{3,3\} & T_3 = t\{3,3\} & Q_2 = P_3 = \{\ \} \times \{3\} \\ S_3 = Y_4^2 = \{3,4\} & T_4 = t\{4,3\} & P_n = \{\ \} \times \{n\}\ (n \geqslant 5) \\ P_4 = \{4,3\} & T_5 = t\{5,3\} & S_n = h\{\ \}s\{n\}\ (n \geqslant 4) \\ I_5 = \{3,5\} & E_5 = r\left\{\begin{matrix}3\\5\end{matrix}\right\} & \\ D_5 = \{5,3\} & & \end{array}$$

Each of the non-uniform regular-faced polyhedra can then be given a concise symbol indicative of its structure. For example, the Miller–Aškinuze solid, the *elongated square gyrobicupola*, is $\mathrm{g}\,Q_4{}^2P_8$.

A convex polyhedron with regular faces is *elementary* if it contains no set of non-adjacent edges forming a regular polygon, i.e., if it cannot be separated by a plane into two smaller convex polyhedra with regular faces. The regular polyhedra $\{n, 3\}$ ($n = 3, 4, 5$), nine of the Archimedean polyhedra, all the prisms, and all the antiprisms except $h\{\ \}s\{3\}$ are elementary. Of the eighty-three non-uniform regular-faced solids obtained from uniform polyhedra, nine are elementary:

$$Y_4,\quad Y_5,\quad Q_3,\quad Q_4,\quad Q_5,\quad R_5,\quad Y_5{}^{-3}I_5,\quad p\text{-}Q_5{}^{-2}E_5,\quad Q_5{}^{-3}E_5.$$

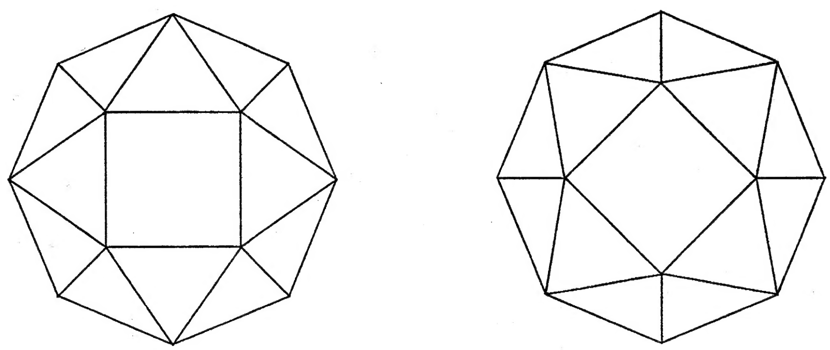
The last three solids result from the respective removal of three pentagonal pyramids from an icosahedron, of two opposite pentagonal cupolas from a rhombicosidodecahedron, and of three pentagonal cupolas from a rhombicosidodecahedron. The *tridiminished icosahedron* $Y_5{}^{-3}I_5$ is the vertex figure of a uniform four-dimensional polytope, the *snub 24-cell* $s\{3, 4, 3\}$ (**4**, p. 163). All of these elementary polyhedra were listed by Zalgaller (**13**, pp. 7–8).

4. Other non-uniform polyhedra. Freudenthal and van der Waerden (**8**) enumerated all the convex polyhedra with congruent regular faces. In addition to the five Platonic solids, these are the *triangular dipyramid* $Y_3{}^2$, the *pentagonal dipyramid* $Y_5{}^2$, the *gyroelongated square dipyramid* $Y_4{}^2S_4$, the *triaugmented triangular prism* $Y_4{}^3P_3$, and one other figure, which they call a "Siamese dodecahedron."

Unlike all the polyhedra discussed so far, this last solid cannot be obtained by taking apart or putting together pieces of uniform polyhedra. It is, however, related to a disphenoid—a tetrahedron regarded as a belt of four triangles between two opposite edges—in the same way that $s\left\{\begin{matrix}3\\4\end{matrix}\right\}$ and $s\left\{\begin{matrix}3\\5\end{matrix}\right\}$ are related to $\left\{\begin{matrix}3\\4\end{matrix}\right\}$ and $\left\{\begin{matrix}3\\5\end{matrix}\right\}$. Consequently, it will be called the *snub disphenoid* and denoted by the symbol $s\,S_2$. From the square antiprism there can be derived in a similar

manner the *snub square antiprism* $s\,S_4$, whose faces consist of 24 triangles and 2 squares. (Since $S_3 = \{3, 4\} = \left\{\begin{matrix}3\\3\end{matrix}\right\}$, $s\,S_3 = s\left\{\begin{matrix}3\\3\end{matrix}\right\} = \{3, 5\}$.)

Snub disphenoid

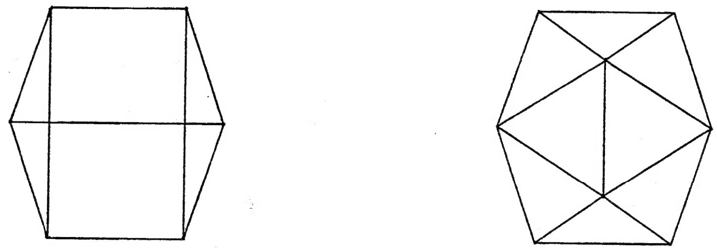


Snub square antiprism

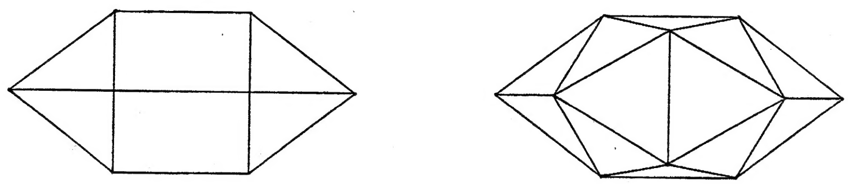
FIGURE 2

Four other convex polyhedra all of whose faces are $\{3\}$'s and $\{4\}$'s are the *sphenocorona* $V_2\,N_2$, the *sphenomegacorona* $V_2\,M_2$, the *hebesphenomegacorona* $U_2\,M_2$, and the *disphenocingulum* $V_2^2G_2$. If we define a *lune* as a complex consisting of two triangles attached to opposite sides of a square, the prefix *spheno-* refers to a wedgelike complex formed by two adjacent lines. The prefix *dispheno-* denotes two such complexes, while *hebespheno-* indicates a blunter complex of two lunes separated by a third lune. The suffix *-corona* refers to a crownlike complex of 8 triangles, and *-megacorona*, to a larger such complex of 12 triangles. The suffix *-cingulum* indicates a belt of 12 triangles.

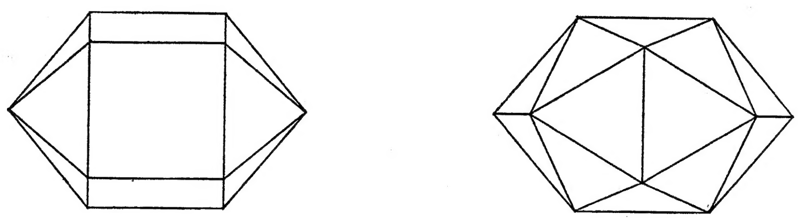
Two polyhedra whose faces include pentagons are the *bilunabirotunda* $L_2^2R_2^2$ and the *triangular hebesphenorotunda* $U_3\,R_3$. The former is regarded as being composed of two lunes and two rotundas—a rotunda here being the complex of faces surrounding a vertex of type $(3\cdot5\cdot3\cdot5)$. The latter is the union of a complex consisting of three lunes separated by a hexagon with



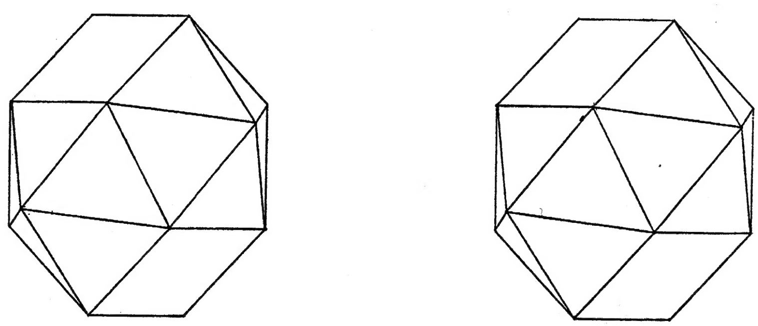
Sphenocorona



Sphenomegacorona



Hebesphenomegacorona



Disphenocingulum

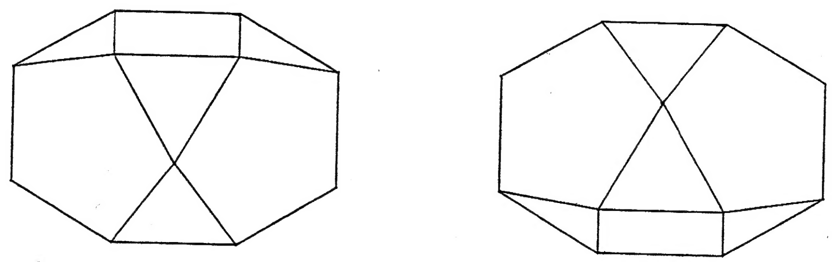
FIGURE 3

triangles attached to its alternate sides and a complex of three triangles and three pentagons surrounding another triangle.

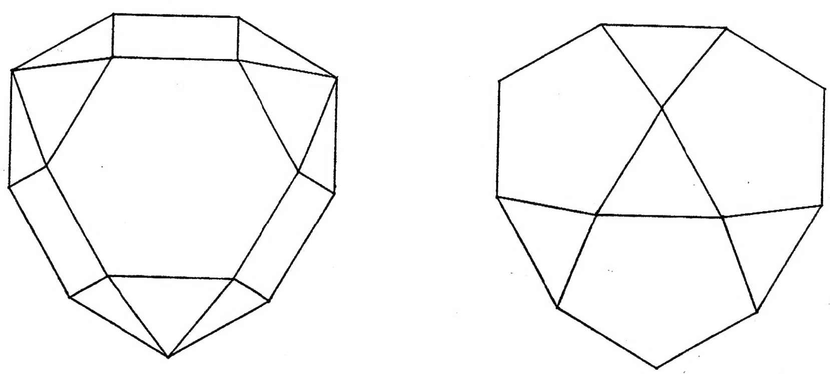
Top and bottom, or front and back, views of each of the polyhedra

$$\mathrm{s\,S_2},\quad \mathrm{s\,S_4},\quad \mathrm{V_2\,N_2},\quad \mathrm{V_2\,M_2},\quad \mathrm{U_2\,M_2},\quad \mathrm{V_2{}^2G_2},\quad \mathrm{L_2{}^2R_2{}^2},\quad \mathrm{U_3\,R_3}$$

are shown in Figures 2, 3, and 4. These solids are all elementary, bringing the number of such non-uniform regular-faced polyhedra up to seventeen. In addition, a non-elementary solid, the *augmented sphenocorona* $\mathrm{Y_4\,V_2\,N_2}$, can be formed by placing a pyramid on one of the square faces of $\mathrm{V_2\,N_2}$.



Bilunabirotunda



Triangular hebesphenorotunda

FIGURE 4

5. Symmetry groups. Coxeter and Moser (**6**, pp. 33–40, 135) have given abstract definitions for each of the finite groups of isometries in E^3. The following is a description of the individual groups, especially as they relate to polyhedra.

The *bilateral* group [ ], of order 2, is generated by a single reflection R, satisfying the relation

$$R^2 = E.$$

The *kaleidoscopic* group $[n] \cong \mathfrak{D}_n$ $(n \geqslant 2)$, of order $2n$, is generated by two reflections, R_1 and R_2, satisfying the relations

$$R_1^2 = R_2^2 = (R_1 R_2)^n = E.$$

The *cyclic* group $[n]^+ \cong \mathfrak{C}_n$ $(n \geqslant 2)$, of order n, is a subgroup of index 2 in $[n]$. It is generated by the rotation $S = R_1 R_2$, satisfying the relation

$$S^n = E.$$

When $n = 1$, the above relations imply that $R_1 = R_2$ and that $S = E$; thus the kaleidoscopic group of order 2 is the bilateral group $[\;] \cong \mathfrak{D}_1$, and the cyclic subgroup is the *identity* group $[\;]^+ \cong \mathfrak{C}_1$. When $n = 2$, the kaleidoscopic group is the direct product of two bilateral groups:

$$[2] \cong [\;] \times [\;].$$

For $n \geqslant 3$, $[n]^+$ is the rotation group of a right regular n-gonal pyramid (other than a regular tetrahedron), and $[n]$ is the complete symmetry group (including reflections).

For integers m and n, $2 \leqslant m \leqslant n$, $(m-2)(n-2) < 4$, the group $[m, n]$, of order

$$4N_1 = \frac{8mn}{4 - (m-2)(n-2)},$$

is generated by three reflections, R_1, R_2, R_3, satisfying the relations

$$R_1^2 = R_2^2 = R_3^2 = (R_1 R_2)^m = (R_2 R_3)^n = (R_1 R_3)^2 = E.$$

For $m = 2$, $N_1 = n$; for $m = 3$, $N_1 = 6n/(6-n)$. The group $[m, n]^+$, of order $2N_1$, is a subgroup of index 2 in $[m, n]$. It is generated by the rotations $S_{12} = R_1 R_2$ and $S_{23} = R_2 R_3$, satisfying the relations

$$S_{12}^m = S_{23}^n = (S_{12} S_{23})^2 = E.$$

The group $[2, 2] \cong [\;] \times [\;] \times [\;]$, of order 8, is generated by reflections R_1, R_2, R_3 in three mutually perpendicular planes. Each reflection generates a subgroup $[\;]$, and each pair of reflections generates a subgroup $[2]$. Each half-turn about a line of intersection of two planes generates a subgroup $[2]^+$. The rotational subgroup $[2, 2]^+ \cong [2]^+ \times [2]^+$ is generated by any two of the three half-turns. A subgroup $[2, 2^+] \cong [\;] \times [2]^+$ is generated by each reflection together with the orthogonal half-turn, e.g., the reflection R_1 and the half-turn $S_{23} = R_2 R_3$, satisfying the relations

$$R_1^2 = S_{23}^2 = (R_1 S_{23})^2 = E.$$

The subgroups $[2]$, $[2, 2^+]$, and $[2, 2]^+$ are isomorphic, and the whole group can be obtained by adjoining to any of them one of the missing reflections:

$$[2, 2] \cong [\;] \times [2] \cong [\;] \times [2, 2^+] \cong [\;] \times [2, 2]^+.$$

The rotatory reflection $Z = R_1 R_2 R_3$ is the *central inversion*, i.e., the "reflection" in the point of intersection of the three planes. The subgroup of [2, 2] generated by Z is the *central* group $[2^+, 2^+] \cong \mathfrak{C}_2$, defined by

$$Z^2 = E.$$

This gives two other ways of expressing [2, 2] as a direct product:

$$[2, 2] \cong [2^+, 2^+] \times [2] \cong [2^+, 2^+] \times [2, 2]^+.$$

Since $R_1 S_{23} = R_1 R_2 R_3$, the group $[2, 2^+]$ also contains the central inversion, and

$$[2, 2^+] \cong [2^+, 2^+] \times [\;] \cong [2^+, 2^+] \times [2]^+.$$

Note that the three groups [ ], $[2]^+$, and $[2^+, 2^+]$, respectively generated by a reflection, a half-turn, and the central inversion, are merely three different geometric representations of the single abstract group of order 2, for which $\mathfrak{D}_1$ and $\mathfrak{C}_2$ are alternative symbols.

The *dihedral* group $[2, n]^+ \cong \mathfrak{D}_n$, of order $2n$, and the extended dihedral group

$$[2, n] \cong [\;] \times [n] \cong [\;] \times [2, n]^+,$$

of order $4n$, are respectively the rotation group and the complete symmetry group of a regular n-gonal prism (other than a cube) for $n \geqslant 3$. When n is even, the group $[2, n]$ contains the central inversion in the form $Z = R_1(R_2 R_3)^{n/2}$, so that

$$[2, n] \cong [2^+, 2^+] \times [n] \cong [2^+, 2^+] \times [2, n]^+ \qquad (n \text{ even}).$$

The group [2, n] contains a subgroup of index 2 generated by the reflection R_1 and the rotation $S_{23} = R_2 R_3$, satisfying the relations

$$R_1^2 = S_{23}^n = E, \qquad R_1 S_{23} = S_{23} R_1.$$

This is the extended cyclic group $[2, n^+] \cong [\;] \times [n]^+$, of order $2n$. The rotational subgroup, generated by S_{23}, is the cyclic group $[n]^+$. When n is even, the group $[2, n^+]$ contains the central inversion in the form $Z = R_1 S_{23}^{n/2}$, and

$$[2, n^+] \cong [2^+, 2^+] \times [n]^+ \qquad (n \text{ even}).$$

The group $[2, 2n]$ contains a subgroup of index 2 generated by the half-turn $S_{12} = R_1 R_2$ and the reflection R_3, satisfying the relations

$$S_{12}^2 = R_3^2 = (S_{12} R_3)^{2n} = E.$$

This is the group $[2^+, 2n] \cong \mathfrak{D}_{2n}$, of order $4n$, the complete symmetry group of a regular n-gonal antiprism (other than a regular octahedron) for $n \geqslant 3$. The rotational subgroup, generated by S_{12} and $R_3 S_{12} R_3$, is $[2, n]^+$. When n is odd, $(S_{12} R_3)^n$ is the central inversion Z, and

$$[2^+, 2n] \cong [2^+, 2^+] \times [n] \cong [2^+, 2^+] \times [2, n]^+ \qquad (n \text{ odd}).$$

The groups $[2^+, 2n]$ and $[2, 2n^+]$ have a common subgroup of index 2, a subgroup of index 4 in $[2, 2n]$, generated by the rotatory reflection

$$T = S_{12} R_3 = R_1 S_{23} = R_1 R_2 R_3,$$

satisfying the relation

$$T^{2n} = E.$$

This is the group $[2^+, 2n^+] \cong \mathfrak{C}_{2n}$, of order $2n$. The rotational subgroup, generated by T^2, is $[n]^+$. When n is odd, T^n is the central inversion Z, and

$$[2^+, 2n^+] \cong [2^+, 2^+] \times [n]^+ \qquad (n \text{ odd}).$$

The groups $[3, 3]^+$, $[3, 4]^+$, and $[3, 5]^+$, of orders 12, 24, and 60, being the rotation groups of the regular polyhedra, are known as the *tetrahedral*, *octahedral*, and *icosahedral* groups. They are isomorphic to symmetric or alternating groups of degree 4 or 5 (**4**, pp. 48–50):

$$[3, 3]^+ \cong \mathfrak{A}_4,$$

$$[3, 4]^+ \cong \mathfrak{S}_4,$$

$$[3, 5]^+ \cong \mathfrak{A}_5.$$

The extended polyhedral groups $[3, 3]$, $[3, 4]$, and $[3, 5]$, of orders 24, 48, and 120, are the complete symmetry groups of the regular polyhedra. The extended tetrahedral group is the symmetric group of degree 4:

$$[3, 3] \cong \mathfrak{S}_4.$$

The extended octahedral group contains the central inversion in the form $Z = (R_1 R_2 R_3)^3$ and is obtained by adjoining this operation to either of the isomorphic groups $[3, 3]$ and $[3, 4]^+$:

$$[3, 4] \cong [2^+, 2^+] \times [3, 3] \cong [2^+, 2^+] \times [3, 4]^+.$$

In the extended icosahedral group the central inversion occurs as $Z = (R_1 R_2 R_3)^5$, and

$$[3, 5] \cong [2^+, 2^+] \times [3, 5]^+.$$

The group $[3, 4]$ contains a subgroup of index 2 generated by the rotation $S_{12} = R_1 R_2$ and the reflection R_3, satisfying the relations

$$S_{12}^3 = R_3^2 = (S_{12}^{-1} R_3 S_{12} R_3)^2 = E.$$

This is the *pyritohedral* group $[3^+, 4]$, of order 24, the complete symmetry group of the crystallographic *pyritohedron* or of the figure consisting of a cube inscribed in a regular dodecahedron. The central inversion occurs in the form $Z = (S_{12} R_3)^3$, while the rotational subgroup, generated by S_{12} and

$R_3 S_{12} R_3$, is $[3, 3]^+$, so that

$$[3^+, 4] \cong [2^+, 2^+] \times [3, 3]^+.$$

The pyritohedral group is also a subgroup of index 5 in [3, 5], generated by the rotation $R_3 R_1 R_2 R_3$ and the reflection R_2.

All the finite three-dimensional symmetry groups are listed in Table I.

TABLE I

FINITE GROUPS OF ISOMETRIES IN E^3

Rotation groups			Extended groups		
Group	Structure	Order	Group	Structure	Order
$[\ ]^+$	$\mathfrak{C}_1$	1	$[\ ]$	$\mathfrak{D}_1$	2
			$[2^+, 2^+]$	$\mathfrak{C}_2$	2
$[n]^+$, $n \geqslant 2$	$\mathfrak{C}_n$	n	$[n]$	$\mathfrak{D}_n$	$2n$
			$[2^+, 2n^+]$	$\mathfrak{C}_{2n}$	$2n$
			$[2, n^+]$	$\mathfrak{D}_1 \times \mathfrak{C}_n$	$2n$
$[2, n]^+$, $n \geqslant 2$	$\mathfrak{D}_n$	$2n$	$[2^+, 2n]$	$\mathfrak{D}_{2n}$	$4n$
			$[2, n]$	$\mathfrak{D}_1 \times \mathfrak{D}_n$	$4n$
$[3, 3]^+$	$\mathfrak{A}_4$	12	$[3, 3]$	$\mathfrak{S}_4$	24
			$[3^+, 4]$	$\mathfrak{C}_2 \times \mathfrak{A}_4$	24
$[3, 4]^+$	$\mathfrak{S}_4$	24	$[3, 4]$	$\mathfrak{C}_2 \times \mathfrak{S}_4$	48
$[3, 5]^+$	$\mathfrak{A}_5$	60	$[3, 5]$	$\mathfrak{C}_2 \times \mathfrak{A}_5$	120

Every rotation, other than the identity, that transforms a solid into itself leaves invariant a unique line, called an *axis of symmetry* of the solid. If the greatest period of any rotation about a given axis is n, the axis is said to be an *n-fold* one. A solid whose rotation group is the identity has no axes. Otherwise, the number of axes of symmetry for each finite rotation group is as follows:

$[n]^+$: 1 n-fold;
$[2, n]^+$: n twofold, 1 n-fold;
$[3, 3]^+$: 3 twofold, 4 threefold;
$[3, 4]^+$: 6 twofold, 4 threefold, 3 fourfold;
$[3, 5]^+$: 15 twofold, 10 threefold, 6 fivefold.

Since these are the only finite rotation groups, it follows that a polyhedron that has more than one threefold, fourfold, or fivefold axis must have tetrahedral, octahedral, or icosahedral symmetry and that no polyhedron can have more than one n-fold axis for $n \geqslant 6$.

Of the convex polyhedra with regular faces, the only ones that have tetrahedral, octahedral, or icosahedral symmetry are the Platonic and Archimedean

solids. The only ones having an axis other than twofold, threefold, fourfold, or fivefold are the n-gonal prisms and antiprisms ($n \geqslant 6$). Thus the rotation group of a non-uniform regular-faced polyhedron is either the identity group or one of the groups $[n]^+$ or $[2, n]^+$ ($n = 2, 3, 4, 5$).

If any of the symmetry operations of a solid are reflections or rotatory reflections, then exactly half of them are, and the rotation group of the solid is a subgroup of index 2 in its complete symmetry group. If not, the rotation group is the whole group, and the solid occurs in two enantiomorphous forms, mirror images of each other. There are seven regular-faced polyhedra of this kind: the Archimedean snub cuboctahedron and snub icosidodecahedron and the non-uniform figures

$$\mathrm{Q}_3{}^2S_6, \quad \mathrm{Q}_4{}^2S_8, \quad \mathrm{Q}_5{}^2\mathrm{S}_{10}, \quad \mathrm{Q}_5\,\mathrm{R}_5\,\mathrm{S}_{10}, \quad \mathrm{R}_5{}^2\mathrm{S}_{10}.$$

On the other hand, four polyhedra have only bilateral symmetry:

$$m\text{-g}\,\mathrm{Q}_5{}^{-1}\mathrm{E}_5, \quad \mathrm{g}^2\mathrm{Q}_5{}^{-1}\mathrm{E}_5, \quad \mathrm{g}\,\mathrm{Q}_5{}^{-2}\mathrm{E}_5, \quad \mathrm{Y}_4\,\mathrm{V}_2\,\mathrm{N}_2.$$

The complete symmetry group of each of the remaining non-uniform solids is one of the groups $[n]$, $[2, n]$, or $[2^+, 2n]$ ($n = 2, 3, 4, 5$). It is remarkable that none of the known convex polyhedra with regular faces is completely asymmetric, i.e., has a symmetry group consisting of the identity alone.

The vertices, edges, and faces of any symmetric polyhedron fall into various equivalence classes. Two vertices, edges, or faces belong to the same equivalence class if there is a symmetry operation of the polyhedron that takes one into the other. The order of the equivalence class to which a particular vertex, edge, or face belongs is equal to the index of the subgroup of the complete symmetry group of the polyhedron that leaves the particular vertex, edge, or face invariant.

The uniform polyhedra are just those regular-faced solids whose vertices all belong to a single equivalence class. Besides the Platonic solids, the only convex polyhedra with regular faces that have all their edges equivalent are the cuboctahedron and the icosidodecahedron, and the only ones whose faces are all equivalent are the triangular and pentagonal dipyramids.

Tables II and III list the different types of faces, edges, and vertices to be found in each convex polyhedron with regular faces. Those edges or vertices that are locally congruent, i.e., edges or vertices of the same type that form equal dihedral angles, are grouped together. In each case the number of faces, edges, or vertices in each equivalence class is indicated in roman type and the number of equivalence classes of the same order in italic type.

Dihedral angles are given to the nearest second. Where minutes and seconds are not shown, the given value is exact.

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TABLE II

CONVEX UNIFORM POLYHEDRA

Name	Symbol	Faces	Edges and dihedral angles	Vertices	Group
Tetrahedron	$\{3, 3\}$	$4\ \{3\}$	$6\ \langle 3\cdot 3\rangle$ $70°\ 31'\ 44''$	$4\ (3^3)$	$[3, 3]$
Octahedron	$\{3, 4\}$	$8\ \{3\}$	$12\ \langle 3\cdot 3\rangle$ $109°\ 28'\ 16''$	$6\ (3^4)$	$[3, 4]$
Cube	$\{4, 3\}$	$6\ \{4\}$	$12\ \langle 4\cdot 4\rangle$ $90°$	$8\ (4^3)$	$[3, 4]$
Icosahedron	$\{3, 5\}$	$20\ \{3\}$	$30\ \langle 3\cdot 3\rangle$ $138°\ 11'\ 23''$	$12\ (3^5)$	$[3, 5]$
Dodecahedron	$\{5, 3\}$	$12\ \{5\}$	$30\ \langle 5\cdot 5\rangle$ $116°\ 33'\ 54''$	$20\ (5^3)$	$[3, 5]$
Cuboctahedron	$\begin{Bmatrix}3\\4\end{Bmatrix}$	$8\ \{3\}$ $6\ \{4\}$	$24\ \langle 3\cdot 4\rangle$ $125°\ 15'\ 52''$	$12\ (3\cdot 4\cdot 3\cdot 4)$	$[3, 4]$
Icosidodecahedron	$\begin{Bmatrix}3\\5\end{Bmatrix}$	$20\ \{3\}$ $12\ \{5\}$	$60\ \langle 3\cdot 5\rangle$ $142°\ 37'\ 21''$	$30\ (3\cdot 5\cdot 3\cdot 5)$	$[3, 5]$
Truncated tetrahedron	$t\{3, 3\}$	$4\ \{3\}$ $4\ \{6\}$	$12\ \langle 3\cdot 6\rangle$ $109°\ 28'\ 16''$ $6\ \langle 6\cdot 6\rangle$ $70°\ 31'\ 44''$	$12\ (3\cdot 6^2)$	$[3, 3]$
Truncated octahedron	$t\{3, 4\}$	$6\ \{4\}$ $8\ \{6\}$	$24\ \langle 4\cdot 6\rangle$ $125°\ 15'\ 52''$ $12\ \langle 6\cdot 6\rangle$ $109°\ 28'\ 16''$	$24\ (4\cdot 6^2)$	$[3, 4]$
Truncated cube	$t\{4, 3\}$	$8\ \{3\}$ $6\ \{8\}$	$24\ \langle 3\cdot 8\rangle$ $125°\ 15'\ 52''$ $12\ \langle 8\cdot 8\rangle$ $90°$	$24\ (3\cdot 8^2)$	$[3, 4]$
Truncated icosahedron	$t\{3, 5\}$	$12\ \{5\}$ $20\ \{6\}$	$60\ \langle 5\cdot 6\rangle$ $142°\ 37'\ 21''$ $30\ \langle 6\cdot 6\rangle$ $138°\ 11'\ 23''$	$60\ (5\cdot 6^2)$	$[3, 5]$
Truncated dodecahedron	$t\{5, 3\}$	$20\ \{3\}$ $12\ \{10\}$	$60\ \langle 3\cdot 10\rangle$ $142°\ 37'\ 21''$ $30\ \langle 10\cdot 10\rangle$ $116°\ 33'\ 54''$	$60\ (3\cdot 10^2)$	$[3, 5]$

TABLE II—*concluded*

Name	Symbol	Faces	Edges and dihedral angles	Vertices	Group
Rhombicuboctahedron	$r\begin{Bmatrix}3\\4\end{Bmatrix}$	8 {3} 12+6 {4}	24 ⟨3·4⟩ 144° 44′ 8″ 24 ⟨4·4⟩ 135°	24 $(3\cdot4^3)$	[3, 4]
Rhombicosidodecahedron	$r\begin{Bmatrix}3\\5\end{Bmatrix}$	20 {3} 30 {4} 12 {5}	60 ⟨3·4⟩ 159° 5′ 41″ 60 ⟨4·5⟩ 148° 16′ 57″	60 (3·4·5·4)	[3, 5]
Truncated cuboctahedron	$t\begin{Bmatrix}3\\4\end{Bmatrix}$	12 {4} 8 {6} 6 {8}	24 ⟨4·6⟩ 144° 44′ 8″ 24 ⟨4·8⟩ 135° 24 ⟨6·8⟩ 125° 15′ 52″	48 (4·6·8)	[3, 4]
Truncated icosidodecahedron	$t\begin{Bmatrix}3\\5\end{Bmatrix}$	30 {4} 20 {6} 12 {10}	60 ⟨4·6⟩ 159° 5′ 41″ 60 ⟨4·10⟩ 148° 16′ 57″ 60 ⟨6·10⟩ 142° 37′ 21″	120 (4·6·10)	[3, 5]
Snub cuboctahedron	$s\begin{Bmatrix}3\\4\end{Bmatrix}$	24+8 {3} 6 {4}	12+24 ⟨3·3⟩ 153° 14′ 5″ 24 ⟨3·4⟩ 142° 59′ 0″	24 $(3^4\cdot4)$	$[3, 4]^+$
Snub icosidodecahedron	$s\begin{Bmatrix}3\\5\end{Bmatrix}$	60+20 {3} 12 {5}	30+60 ⟨3·3⟩ 164° 10′ 31″ 60 ⟨3·5⟩ 152° 55′ 48″	60 $(3^4\cdot5)$	$[3, 5]^+$
n-gonal prism ($n = 3, 5, 6, \ldots$)	$\{\ \} \times \{n\}$	n {4} 2 $\{n\}$	n ⟨4·4⟩ $180°\cdot(n-2)/n$ $2n$ $\langle 4\cdot n\rangle$ 90°	$2n$ $(4^2\cdot n)$	$[2, n]$
n-gonal antiprism ($n = 4, 5, 6, \ldots$)	$h\{\ \}s\{n\}$	$2n$ {3} 2 $\{n\}$	$2n$ ⟨3·3⟩ $2\tan^{-1}\frac{1}{2}a_n$* $2n$ $\langle 3\cdot n\rangle$ $180° - \tan^{-1} a_n$	$2n$ $(3^3\cdot n)$	$[2^+, 2n]$

*$a_n = \sqrt{(3\cot^2 \pi/2n - 1)}$.

TABLE III

NON-UNIFORM CONVEX POLYHEDRA WITH REGULAR FACES

No.	Name	Symbol	Faces	Edges and dihedral angles		Vertices	Group
1	Square pyramid	Y_4	$4\ \{3\}$ $1\ \{4\}$	$4\ \langle 3\cdot 3\rangle$ $4\ \langle 3\cdot 4\rangle$	109° 28′ 16″ 54° 44′ 8″	$4\ (3^2\cdot 4)$ $1\ (3^4)$	[4]
2	Pentagonal pyramid	Y_5	$5\ \{3\}$ $1\ \{5\}$	$5\ \langle 3\cdot 3\rangle$ $5\ \langle 3\cdot 5\rangle$	138° 11′ 23″ 37° 22′ 39″	$5\ (3^2\cdot 5)$ $1\ (3^5)$	[5]
3	Triangular cupola	Q_3	$1+3\ \{3\}$ $3\ \{4\}$ $1\ \{6\}$	$3+6\ \langle 3\cdot 4\rangle$ $3\ \langle 3\cdot 6\rangle$ $3\ \langle 4\cdot 6\rangle$	125° 15′ 52″ 70° 31′ 44″ 54° 44′ 8″	$6\ (3\cdot 4\cdot 6)$ $3\ (3\cdot 4\cdot 3\cdot 4)$	[3]
4	Square cupola	Q_4	$4\ \{3\}$ $1+4\ \{4\}$ $1\ \{8\}$	$8\ \langle 3\cdot 4\rangle$ $4\ \langle 4\cdot 4\rangle$ $4\ \langle 3\cdot 8\rangle$ $4\ \langle 4\cdot 8\rangle$	144° 44′ 8″ 135° 54° 44′ 8″ 45°	$8\ (3\cdot 4\cdot 8)$ $4\ (3\cdot 4^3)$	[4]
5	Pentagonal cupola	Q_5	$5\ \{3\}$ $5\ \{4\}$ $1\ \{5\}$ $1\ \{10\}$	$10\ \langle 3\cdot 4\rangle$ $5\ \langle 4\cdot 5\rangle$ $5\ \langle 3\cdot 10\rangle$ $5\ \langle 4\cdot 10\rangle$	159° 5′ 41″ 148° 16′ 57″ 37° 22′ 39″ 31° 43′ 3″	$10\ (3\cdot 4\cdot 10)$ $5\ (3\cdot 4\cdot 5\cdot 4)$	[5]
6	Pentagonal rotunda	R_5	$2\cdot 5\ \{3\}$ $1+5\ \{5\}$ $1\ \{10\}$	$5+2\cdot 10\ \langle 3\cdot 5\rangle$ $5\ \langle 3\cdot 10\rangle$ $5\ \langle 5\cdot 10\rangle$	142° 37′ 21″ 79° 11′ 16″ 63° 26′ 6″	$10\ (3\cdot 5\cdot 10)$ $2\cdot 5\ (3\cdot 5\cdot 3\cdot 5)$	[5]
7	Elongated triangular pyramid	$Y_3 P_3$	$1+3\ \{3\}$ $3\ \{4\}$	$3\ \langle 3\cdot 3\rangle$ $3\ \langle 3\cdot 4\rangle$ $3\ \langle 3\cdot 4\rangle$ $3\ \langle 4\cdot 4\rangle$	70° 31′ 44″ 160° 31′ 44″ 90° 60°	$1\ (3^3)$ $3\ (3\cdot 4^2)$ $3\ (3^2\cdot 4^2)$	[3]
8	Elongated square pyramid	$Y_4 P_4$	$4\ \{3\}$ $1+4\ \{4\}$	$4\ \langle 3\cdot 3\rangle$ $4\ \langle 3\cdot 4\rangle$ $2\cdot 4\ \langle 4\cdot 4\rangle$	109° 28′ 16″ 144° 44′ 8″ 90°	$4\ (4^3)$ $1\ (3^4)$ $4\ (3^2\cdot 4^2)$	[4]

TABLE III—*continued*

No.	Name	Symbol	Faces	Edges and dihedral angles	Vertices	Group
9	Elongated pentagonal pyramid	$Y_5 P_5$	5 {3} 5 {4} 1 {5}	5 ⟨3·3⟩ 138° 11′ 23″ 5 ⟨3·4⟩ 127° 22′ 39″ 5 ⟨4·4⟩ 108° 5 ⟨4·5⟩ 90°	1 (3^5) 5 $(4^2 \cdot 5)$ 5 $(3^2 \cdot 4^2)$	[5]
10	Gyroelongated square pyramid	$Y_4 S_4$	3·4 {3} 1 {4}	4 ⟨3·3⟩ 158° 34′ 18″ 8 ⟨3·3⟩ 127° 33′ 6″ 4 ⟨3·3⟩ 109° 28′ 16″ 4 ⟨3·4⟩ 103° 50′ 10″	1 (3^4) 4 $(3^3 \cdot 4)$ 4 (3^5)	[4]
11	Gyroelongated pentagonal pyramid	$Y_5 S_5$	3·5 {3} 1 {5}	2·5+10 ⟨3·3⟩ 138° 11′ 23″ 5 ⟨3·5⟩ 100° 48′ 44″	5 $(3^3 \cdot 5)$ 1+5 (3^5)	[5]
12	Triangular dipyramid	Y_3^2	6 {3}	3 ⟨3·3⟩ 141° 3′ 27″ 6 ⟨3·3⟩ 70° 31′ 44″	2 (3^3) 3 (3^4)	[2, 3]
13	Pentagonal dipyramid	Y_5^2	10 {3}	10 ⟨3·3⟩ 138° 11′ 23″ 5 ⟨3·3⟩ 74° 45′ 17″	5 (3^4) 2 (3^5)	[2, 5]
14	Elongated triangular dipyramid	$Y_3^2 P_3$	6 {3} 3 {4}	6 ⟨3·3⟩ 70° 31′ 44″ 6 ⟨3·4⟩ 160° 31′ 44″ 3 ⟨4·4⟩ 60°	2 (3^3) 6 $(3^2 \cdot 4^2)$	[2, 3]
15	Elongated square dipyramid	$Y_4^2 P_4$	8 {3} 4 {4}	8 ⟨3·3⟩ 109° 28′ 16″ 8 ⟨3·4⟩ 144° 44′ 8″ 4 ⟨4·4⟩ 90°	2 (3^4) 8 $(3^2 \cdot 4^2)$	[2, 4]
16	Elongated pentagonal dipyramid	$Y_5^2 P_5$	10 {3} 5 {4}	10 ⟨3·3⟩ 138° 11′ 23″ 10 ⟨3·4⟩ 127° 22′ 39″ 5 ⟨4·4⟩ 108°	10 $(3^2 \cdot 4^2)$ 2 (3^5)	[2, 5]
17	Gyroelongated square dipyramid	$Y_4^2 S_4$	2·8 {3}	8 ⟨3·3⟩ 158° 34′ 18″ 8 ⟨3·3⟩ 127° 33′ 6″ 8 ⟨3·3⟩ 109° 28′ 16″	2 (3^4) 8 (3^5)	$[2^+, 8]$

18	Elongated triangular cupola	$Q_3 P_6$	1+3 {3} 3·3 {4} 1 {6}	3 ⟨3·4⟩ 3+6 ⟨3·4⟩ 3 ⟨4·4⟩ 6 ⟨4·4⟩ 2·3 ⟨4·6⟩	160° 31′ 44″ 125° 15′ 52″ 144° 44′ 8″ 120° 90°	6 $(4^2\cdot 6)$ 3 $(3\cdot 4\cdot 3\cdot 4)$ 6 $(3\cdot 4^3)$	[3]
19	Elongated square cupola	$Q_4 P_8$	4 {3} 1+3·4 {4} 1 {8}	4+8 ⟨3·4⟩ 2·4+8 ⟨4·4⟩ 2·4 ⟨4·8⟩	144° 44′ 8″ 135° 90°	8 $(4^2\cdot 8)$ 4+8 $(3\cdot 4^3)$	[4]
20	Elongated pentagonal cupola	$Q_5 P_{10}$	5 {3} 3·5 {4} 1 {5} 1 {10}	10 ⟨3·4⟩ 5 ⟨3·4⟩ 10 ⟨4·4⟩ 5 ⟨4·4⟩ 5 ⟨4·5⟩ 2·5 ⟨4·10⟩	159° 5′ 41″ 127° 22′ 39″ 144° 121° 43′ 3″ 148° 16′ 57″ 90°	10 $(4^2\cdot 10)$ 10 $(3\cdot 4^3)$ 5 $(3\cdot 4\cdot 5\cdot 4)$	[5]
21	Elongated pentagonal rotunda	$R_5 P_{10}$	2·5 {3} 2·5 {4} 1+5 {5} 1 {10}	5 ⟨3·4⟩ 10 ⟨4·4⟩ 5+2·10 ⟨3·5⟩ 5 ⟨4·5⟩ 2·5 ⟨4·10⟩	169° 11′ 16″ 144° 142° 37′ 21″ 153° 26′ 6″ 90°	10 $(4^2\cdot 10)$ 10 $(3\cdot 4^2\cdot 5)$ 2·5 $(3\cdot 5\cdot 3\cdot 5)$	[5]
22	Gyroelongated triangular cupola	$Q_3 S_6$	1+3·3+6 {3} 3 {4} 1 {6}	3 ⟨3·3⟩ 2·6 ⟨3·3⟩ 3 ⟨3·4⟩ 3+6 ⟨3·4⟩ 6 ⟨3·6⟩	169° 25′ 42″ 145° 13′ 19″ 153° 38′ 6″ 125° 15′ 52″ 98° 53′ 58″	3 $(3\cdot 4\cdot 3\cdot 4)$ 2·3 $(3^3\cdot 6)$ 6 $(3^4\cdot 4)$	[3]
23	Gyroelongated square cupola	$Q_4 S_8$	3·4+8 {3} 1+4 {4} 1 {8}	2·8 ⟨3·3⟩ 4 ⟨3·3⟩ 8 ⟨3·4⟩ 4 ⟨3·4⟩ 4 ⟨4·4⟩ 8 ⟨3·8⟩	153° 57′ 45″ 151° 19′ 48″ 144° 44′ 8″ 141° 35′ 40″ 135° 96° 35′ 40″	4 $(3\cdot 4^3)$ 2·4 $(3^3\cdot 8)$ 8 $(3^4\cdot 4)$	[4]

TABLE III—*continued*

No.	Name	Symbol	Faces	Edges and dihedral angles		Vertices	Group
24	Gyroelongated pentagonal cupola	$Q_5 S_{10}$	$3\cdot5+10$ {3} 5 {4} 1 {5} 1 {10}	$2\cdot10$ ⟨3·3⟩ 5 ⟨3·3⟩ 10 ⟨3·4⟩ 5 ⟨3·4⟩ 5 ⟨4·5⟩ 10 ⟨3·10⟩	159° 11′ 11″ 132° 37′ 26″ 159° 5′ 41″ 126° 57′ 51″ 148° 16′ 57″ 95° 14′ 48″	5 (3·4·5·4) $2\cdot5$ (3^3·10) 10 (3^4·4)	[5]
25	Gyroelongated pentagonal rotunda	$R_5 S_{10}$	$4\cdot5+10$ {3} 1+5 {5} 1 {10}	5 ⟨3·3⟩ $2\cdot10$ ⟨3·3⟩ 5 ⟨3·5⟩ $5+2\cdot10$ ⟨3·5⟩ 10 ⟨3·10⟩	174° 26′ 4″ 159° 11′ 11″ 158° 40′ 54″ 142° 37′ 21″ 95° 14′ 48″	$2\cdot5$ (3·5·3·5) $2\cdot5$ (3^3·10) 10 (3^4·5)	[5]
26	Gyrobifastigium	g Q_2^2	4 {3} 4 {4}	4 ⟨3·4⟩ 8 ⟨3·4⟩ 2 ⟨4·4⟩	150° 90° 60°	4 (3·4^2) 4 (3·4·3·4)	[2^+, 4]
27	Triangular orthobicupola	Q_3^2	2+6 {3} 6 {4}	3 ⟨3·3⟩ 6+12 ⟨3·4⟩ 3 ⟨4·4⟩	141° 3′ 27″ 125° 15′ 52″ 109° 28′ 16″	6 (3^2·4^2) 6 (3·4·3·4)	[2, 3]
28	Square orthobicupola	Q_4^2	8 {3} 2+8 {4}	4 ⟨3·3⟩ 16 ⟨3·4⟩ 8 ⟨4·4⟩ 4 ⟨4·4⟩	109° 28′ 16″ 144° 44′ 8″ 135° 90°	8 (3^2·4^2) 8 (3·4^3)	[2, 4]
29	Square gyrobicupola	g Q_4^2	8 {3} 2+8 {4}	16 ⟨3·4⟩ 8 ⟨3·4⟩ 8 ⟨4·4⟩	144° 44′ 8″ 99° 44′ 8″ 135°	8 (3·4·3·4) 8 (3·4^3)	[2^+, 8]
30	Pentagonal orthobicupola	Q_5^2	10 {3} 10 {4} 2 {5}	5 ⟨3·3⟩ 20 ⟨3·4⟩ 5 ⟨4·4⟩ 10 ⟨4·5⟩	74° 45′ 17″ 159° 5′ 41″ 63° 26′ 6″ 148° 16′ 57″	10 (3^2·4^2) 10 (3·4·5·4)	[2, 5]

31	Pentagonal gyrobicupola	g Q_5^2	10 {3} 10 {4} 2 {5}	20 ⟨3·4⟩ 10 ⟨3·4⟩ 10 ⟨4·5⟩	159° 5′ 41″ 69° 5′ 41″ 148° 16′ 57″	10 (3·4·3·4) 10 (3·4·5·4)	[2^+, 10]
32	Pentagonal orthocupolarotunda	$Q_5 R_5$	*3*·5 {3} 5 {4} *2*+5 {5}	10 ⟨3·4⟩ 5 ⟨3·4⟩ 5+*2*·10 ⟨3·5⟩ 5 ⟨3·5⟩ 5 ⟨4·5⟩	159° 5′ 41″ 110° 54′ 19″ 142° 37′ 21″ 100° 48′ 44″ 148° 16′ 57″	10 (3·4·3·5) 5 (3·4·5·4) *2*·5 (3·5·3·5)	[5]
33	Pentagonal gyrocupolarotunda	g $Q_5 R_5$	*3*·5 {3} 5 {4} 2+5 {5}	5 ⟨3·3⟩ 10 ⟨3·4⟩ 5+*2*·10 ⟨3·5⟩ 5 ⟨4·5⟩ 5 ⟨4·5⟩	116° 33′ 54″ 159° 5′ 41″ 142° 37′ 21″ 148° 16′ 57″ 95° 9′ 9″	10 $(3^2\cdot4\cdot5)$ 5 (3·4·5·4) *2*·5 (3·5·3·5)	[5]
34	Pentagonal orthobirotunda	R_5^2	*2*·10 {3} 2+10 {5}	5 ⟨3·3⟩ 10+*2*·20 ⟨3·5⟩ 5 ⟨5·5⟩	158° 22′ 31″ 142° 37′ 21″ 126° 52′ 12″	10 $(3^2\cdot5^2)$ *2*·10 (3·5·3·5)	[2, 5]
35	Elongated triangular orthobicupola	$Q_3^2P_6$	2+6 {3} *2*·3+6 {4}	6 ⟨3·4⟩ 6+12 ⟨3·4⟩ 6 ⟨4·4⟩ 6 ⟨4·4⟩	160° 31′ 44″ 125° 15′ 52″ 144° 44′ 8″ 120°	6 (3·4·3·4) 12 $(3\cdot4^3)$	[2, 3]
36	Elongated triangular gyrobicupola	g $Q_3^2P_6$	2+6 {3} *2*·6 {4}	6 ⟨3·4⟩ 6+12 ⟨3·4⟩ 6 ⟨4·4⟩ 6 ⟨4·4⟩	160° 31′ 44″ 125° 15′ 52″ 144° 44′ 8″ 120°	6 (3·4·3·4) 12 $(3\cdot4^3)$	[2^+, 6]
37	Elongated square gyrobicupola	g $Q_4^2P_8$	8 {3} 2+*2*·8 {4}	8+16 ⟨3·4⟩ *3*·8 ⟨4·4⟩	144° 44′ 8″ 135°	8+16 $(3\cdot4^3)$	[2^+, 8]
38	Elongated pentagonal orthobicupola	$Q_5^2P_{10}$	10 {3} *2*·5+10 {4} 2 {5}	20 ⟨3·4⟩ 10 ⟨3·4⟩ 10 ⟨4·4⟩ 10 ⟨4·4⟩ 10 ⟨4·5⟩	159° 5′ 41″ 127° 22′ 39″ 144° 121° 43′ 3″ 148° 16′ 57″	20 $(3\cdot4^3)$ 10 (3·4·5·4)	[2, 5]

TABLE III—*continued*

No.	Name	Symbol	Faces	Edges and dihedral angles	Vertices	Group
39	Elongated pentagonal gyrobicupola	$g\ Q_5{}^2P_{10}$	10 {3} *2*·10 {4} 2 {5}	20 ⟨3·4⟩ 159° 5′ 41″ 10 ⟨3·4⟩ 127° 22′ 39″ 10 ⟨4·4⟩ 144° 10 ⟨4·4⟩ 121° 43′ 3″ 10 ⟨4·5⟩ 148° 16′ 57″	20 $(3\cdot 4^3)$ 10 $(3\cdot 4\cdot 5\cdot 4)$	$[2^+, 10]$
40	Elongated pentagonal orthocupolarotunda	$Q_5\ R_5\ P_{10}$	*3*·5 {3} *3*·5 {4} *2*+5 {5}	5 ⟨3·4⟩ 169° 11′ 16″ 10 ⟨3·4⟩ 159° 5′ 41″ 5 ⟨3·4⟩ 127° 22′ 39″ 10 ⟨4·4⟩ 144° 5 ⟨4·4⟩ 121° 43′ 3″ 5+*2*·10 ⟨3·5⟩ 142° 37′ 21″ 5 ⟨4·5⟩ 153° 26′ 6″ 5 ⟨4·5⟩ 148° 16′ 57″	10 $(3\cdot 4^3)$ 10 $(3\cdot 4^2\cdot 5)$ 5 $(3\cdot 4\cdot 5\cdot 4)$ *2*·5 $(3\cdot 5\cdot 3\cdot 5)$	[5]
41	Elongated pentagonal gyrocupolarotunda	$g\ Q_5\ R_5\ P_{10}$	*3*·5 {3} *3*·5 {4} *2*+5 {5}	5 ⟨3·4⟩ 169° 11′ 16″ 10 ⟨3·4⟩ 159° 5′ 41″ 5 ⟨3·4⟩ 127° 22′ 39″ 10 ⟨4·4⟩ 144° 5 ⟨4·4⟩ 121° 43′ 3″ 5+*2*·10 ⟨3·5⟩ 142° 37′ 21″ 5 ⟨4·5⟩ 153° 26′ 6″ 5 ⟨4·5⟩ 148° 16′ 57″	10 $(3\cdot 4^3)$ 10 $(3\cdot 4^2\cdot 5)$ 5 $(3\cdot 4\cdot 5\cdot 4)$ *2*·5 $(3\cdot 5\cdot 3\cdot 5)$	[5]
42	Elongated pentagonal orthobirotunda	$R_5{}^2P_{10}$	*2*·10 {3} *2*·5 {4} 2+10 {5}	10 ⟨3·4⟩ 169° 11′ 16″ 10 ⟨4·4⟩ 144° 10+*2*·20 ⟨3·5⟩ 142° 37′ 21″ 10 ⟨4·5⟩ 153° 26′ 6″	20 $(3\cdot 4^2\cdot 5)$ *2*·10 $(3\cdot 5\cdot 3\cdot 5)$	[2, 5]

43	Elongated pentagonal gyrobirotunda	g $R_5^2P_{10}$	*2*·10 {3} 10 {4} 2+10 {5}	10 ⟨3·4⟩ 10 ⟨4·4⟩ 10+*2*·20 ⟨3·5⟩ 10 ⟨4·5⟩	169° 11′ 16″ 144° 142° 37′ 21″ 153° 26′ 6″	20 (3·4²·5) *2*·10 (3·5·3·5)	[2⁺, 10]
44	Gyroelongated triangular bicupola	$Q_3^2S_6$	2+*3*·6 {3} 6 {4}	6 ⟨3·3⟩ *2*·3+6 ⟨3·3⟩ 6 ⟨3·4⟩ *3*·6 ⟨3·4⟩	169° 25′ 42″ 145° 13′ 19″ 153° 38′ 6″ 125° 15′ 52″	6 (3·4·3·4) *2*·6 (3⁴·4)	[2, 3]⁺
45	Gyroelongated square bicupola	$Q_4^2S_8$	*3*·8 {3} 2+8 {4}	*2*·4+8 ⟨3·3⟩ 8 ⟨3·3⟩ *2*·8 ⟨3·4⟩ 8 ⟨3·4⟩ 8 ⟨4·4⟩	153° 57′ 45″ 151° 19′ 48″ 144° 44′ 8″ 141° 35′ 40″ 135°	8 (3·4³) *2*·8 (3⁴·4)	[2, 4]⁺
46	Gyroelongated pentagonal bicupola	$Q_5^2S_{10}$	*3*·10 {3} 10 {4} 2 {5}	*2*·5+10 ⟨3·3⟩ 10 ⟨3·3⟩ *2*·10 ⟨3·4⟩ 10 ⟨3·4⟩ 10 ⟨4·5⟩	159° 11′ 11″ 132° 37′ 26″ 159° 5′ 41″ 126° 57′ 51″ 148° 16′ 57″	10 (3·4·5·4) *2*·10 (3⁴·4)	[2, 5]⁺
47	Gyroelongated pentagonal cupolarotunda	$Q_5\,R_5\,S_{10}$	*7*·5 {3} 5 {4} *2*+5 {5}	5 ⟨3·3⟩ *4*·5 ⟨3·3⟩ 5 ⟨3·3⟩ *2*·5 ⟨3·4⟩ 5 ⟨3·4⟩ 5 ⟨3·5⟩ *5*·5 ⟨3·5⟩ 5 ⟨4·5⟩	174° 26′ 4″ 159° 11′ 11″ 132° 37′ 26″ 159° 5′ 41″ 126° 57′ 51″ 158° 40′ 54″ 142° 37′ 21″ 148° 16′ 57″	5 (3·4·5·4) *2*·5 (3·5·3·5) *2*·5 (3⁴·4) *2*·5 (3⁴·5)	[5]⁺
48	Gyroelongated pentagonal birotunda	$R_5^2S_{10}$	*4*·10 {3} 2+10 {5}	10 ⟨3·3⟩ *2*·5+10 ⟨3·3⟩ 10 ⟨3·5⟩ *5*·10 ⟨3·5⟩	174° 26′ 4″ 159° 11′ 11″ 158° 40′ 54″ 142° 37′ 21″	*2*·10 (3·5·3·5) *2*·10 (3⁴·5)	[2, 5]⁺

TABLE III—*continued*

No.	Name	Symbol	Faces	Edges and dihedral angles		Vertices	Group
49	Augmented triangular prism	$Y_4\,P_3$	*3*·2 {3} 2 {4}	2 ⟨3·3⟩ 4 ⟨3·3⟩ 2 ⟨3·4⟩ 4 ⟨3·4⟩ 1 ⟨4·4⟩	144° 44′ 8″ 109° 28′ 16″ 114° 44′ 8″ 90° 60°	2 $(3\cdot4^2)$ 1 (3^4) 4 $(3^3\cdot4)$	[2]
50	Biaugmented triangular prism	$Y_4^2P_3$	*3*·2+4 {3} 1 {4}	1 ⟨3·3⟩ 4 ⟨3·3⟩ *2*·4 ⟨3·3⟩ 2 ⟨3·4⟩ 2 ⟨3·4⟩	169° 28′ 16″ 144° 44′ 8″ 109° 28′ 16″ 114° 44′ 8″ 90°	2 (3^4) 4 $(3^3\cdot4)$ 2 (3^5)	[2]
51	Triaugmented triangular prism	$Y_4^3P_3$	2+*2*·6 {3}	3 ⟨3·3⟩ 6 ⟨3·3⟩ 12 ⟨3·3⟩	169° 28′ 16″ 144° 44′ 8″ 109° 28′ 16″	3 (3^4) 6 (3^5)	[2, 3]
52	Augmented pentagonal prism	$Y_4\,P_5$	*2*·2 {3} *2*·2 {4} 2 {5}	4 ⟨3·3⟩ 2 ⟨3·4⟩ 1+2 ⟨4·4⟩ 2 ⟨3·5⟩ *2*·4 ⟨4·5⟩	109° 28′ 16″ 162° 44′ 8″ 108° 144° 44′ 8″ 90°	2+4 $(4^2\cdot5)$ 1 (3^4) 4 $(3^2\cdot4\cdot5)$	[2]
53	Biaugmented pentagonal prism	$Y_4^2P_5$	*2*·2+4 {3} 1+2 {4} 2 {5}	*2*·4 ⟨3·3⟩ *2*·2 ⟨3·4⟩ 1 ⟨4·4⟩ 4 ⟨3·5⟩ 2+4 ⟨4·5⟩	109° 28′ 16″ 162° 44′ 8″ 108° 144° 44′ 8″ 90°	2 $(4^2\cdot5)$ 2 (3^4) *2*·4 $(3^2\cdot4\cdot5)$	[2]
54	Augmented hexagonal prism	$Y_4\,P_6$	*2*·2 {3} 1+*2*·2 {4} 2 {6}	4 ⟨3·3⟩ 2 ⟨3·4⟩ *2*·2 ⟨4·4⟩ 2 ⟨3·6⟩ 2+*2*·4 ⟨4·6⟩	109° 28′ 16″ 174° 44′ 8″ 120° 144° 44′ 8″ 90°	*2*·4 $(4^2\cdot6)$ 1 (3^4) 4 $(3^2\cdot4\cdot6)$	[2]

55	Parabiaugmented hexagonal prism	*p*-$Y_4^2P_6$	*2*·4 {3} 4 {4} 2 {6}	8 ⟨3·3⟩ 4 ⟨3·4⟩ 2 ⟨4·4⟩ 4 ⟨3·6⟩ 8 ⟨4·6⟩	109° 28′ 16″ 174° 44′ 8″ 120° 144° 44′ 8″ 90°	4 $(4^2 \cdot 6)$ 2 (3^4) 8 $(3^2 \cdot 4 \cdot 6)$	[2, 2]
56	Metabiaugmented hexagonal prism	*m*-$Y_4^2P_6$	*2*·2+4 {3} *2*+2 {4} 2 {6}	*2*·4 ⟨3·3⟩ *2*·2 ⟨3·4⟩ 2 ⟨4·4⟩ 4 ⟨3·6⟩ *2*·2+4 ⟨4·6⟩	109° 28′ 16″ 174° 44′ 8″ 120° 144° 44′ 8″ 90°	4 $(4^2 \cdot 6)$ 2 (3^4) *2*·4 $(3^2 \cdot 4 \cdot 6)$	[2]
57	Triaugmented hexagonal prism	$Y_4^3P_6$	*2*·6 {3} 3 {4} 2 {6}	12 ⟨3·3⟩ 6 ⟨3·4⟩ 6 ⟨3·6⟩ 6 ⟨4·6⟩	109° 28′ 16″ 174° 44′ 8″ 144° 44′ 8″ 90°	3 (3^4) 12 $(3^2 \cdot 4 \cdot 6)$	[2, 3]
58	Augmented dodecahedron	$Y_5 D_5$	5 {3} 1+*2*·5 {5}	5 ⟨3·3⟩ 5 ⟨3·5⟩ *3*·5+10 ⟨5·5⟩	138° 11′ 23″ 153° 56′ 33″ 116° 33′ 54″	*3*·5 (5^3) 5 $(3^2 \cdot 5^2)$ 1 (3^5)	[5]
59	Parabiaugmented dodecahedron	*p*-$Y_5^2D_5$	10 {3} 10 {5}	10 ⟨3·3⟩ 10 ⟨3·5⟩ *2*·10 ⟨5·5⟩	138° 11′ 23″ 153° 56′ 33″ 116° 33′ 54″	10 (5^3) 10 $(3^2 \cdot 5^2)$ 2 (3^5)	[2⁺, 10]
60	Metabiaugmented dodecahedron	*m*-$Y_5^2D_5$	2+*2*·4 {3} *3*·2+4 {5}	2+*2*·4 ⟨3·3⟩ 2+*2*·4 ⟨3·5⟩ *2*+2+*4*·4 ⟨5·5⟩	138° 11′ 23″ 153° 56′ 33″ 116° 33′ 54″	*3*·2+4 (5^3) 2+*2*·4 $(3^2 \cdot 5^2)$ 2 (3^5)	[2]
61	Triaugmented dodecahedron	$Y_5^3D_5$	3+*2*·6 {3} *3*·3 {5}	3+*2*·6 ⟨3·3⟩ 3+*2*·6 ⟨3·5⟩ *3*·3+6 ⟨5·5⟩	138° 11′ 23″ 153° 56′ 33″ 116° 33′ 54″	*2*+3 (5^3) 3+*2*·6 $(3^2 \cdot 5^2)$ 3 (3^5)	[3]
62	Metabidiminished icosahedron	*m*-$Y_5^{-2}I_5$	*3*·2+4 {3} 2 {5}	1+2+*2*·4 ⟨3·3⟩ *2*·4 ⟨3·5⟩ 1 ⟨5·5⟩	138° 11′ 23″ 100° 48′ 44″ 63° 26′ 6″	2 $(3 \cdot 5^2)$ 2+4 $(3^3 \cdot 5)$ 2 (3^5)	[2]

TABLE III—*continued*

No.	Name	Symbol	Faces	Edges and dihedral angles		Vertices	Group
63	Tridiminished icosahedron	$Y_5{}^{-3}I_5$	$2+3\ \{3\}$ $3\ \{5\}$	$3\ \langle 3\cdot3\rangle$ $3+6\ \langle 3\cdot5\rangle$ $3\ \langle 5\cdot5\rangle$	138° 11′ 23″ 100° 48′ 44″ 63° 26′ 6″	$2\cdot3\ (3\cdot5^2)$ $3\ (3^3\cdot5)$	[3]
64	Augmented tridiminished icosahedron	$Y_3\,Y_5{}^{-3}I_5$	$1+2\cdot3\ \{3\}$ $3\ \{5\}$	$3\ \langle 3\cdot3\rangle$ $3\ \langle 3\cdot3\rangle$ $3\ \langle 3\cdot3\rangle$ $6\ \langle 3\cdot5\rangle$ $3\ \langle 5\cdot5\rangle$	171° 20′ 28″ 138° 11′ 23″ 70° 31′ 44″ 100° 48′ 44″ 63° 26′ 6″	$1\ (3^3)$ $3\ (3\cdot5^2)$ $3\ (3^3\cdot5)$ $3\ (3^2\cdot5^2)$	[3]
65	Augmented truncated tetrahedron	$Q_3\,T_3$	$2+2\cdot3\ \{3\}$ $3\ \{4\}$ $3\ \{6\}$	$3\ \langle 3\cdot4\rangle$ $3+6\ \langle 3\cdot4\rangle$ $3\ \langle 3\cdot6\rangle$ $3+6\ \langle 3\cdot6\rangle$ $3\ \langle 6\cdot6\rangle$	164° 12′ 25″ 125° 15′ 52″ 141° 3′ 27″ 109° 28′ 16″ 70° 31′ 44″	$2\cdot3\ (3\cdot6^2)$ $3\ (3\cdot4\cdot3\cdot4)$ $6\ (3\cdot4\cdot3\cdot6)$	[3]
66	Augmented truncated cube	$Q_4\,T_4$	$3\cdot4\ \{3\}$ $1+4\ \{4\}$ $1+4\ \{8\}$	$4\ \langle 3\cdot4\rangle$ $8\ \langle 3\cdot4\rangle$ $4\ \langle 4\cdot4\rangle$ $4\ \langle 3\cdot8\rangle$ $4+2\cdot8\ \langle 3\cdot8\rangle$ $2\cdot4\ \langle 8\cdot8\rangle$	170° 15′ 52″ 144° 44′ 8″ 135° 144° 44′ 8″ 125° 15′ 52″ 90°	$2\cdot4+8\ (3\cdot8^2)$ $4\ (3\cdot4^3)$ $8\ (3\cdot4\cdot3\cdot8)$	[4]
67	Biaugmented truncated cube	$Q_4{}^2T_4$	$2\cdot8\ \{3\}$ $2+8\ \{4\}$ $4\ \{8\}$	$8\ \langle 3\cdot4\rangle$ $16\ \langle 3\cdot4\rangle$ $8\ \langle 4\cdot4\rangle$ $8\ \langle 3\cdot8\rangle$ $16\ \langle 3\cdot8\rangle$ $4\ \langle 8\cdot8\rangle$	170° 15′ 52″ 144° 44′ 8″ 135° 144° 44′ 8″ 125° 15′ 52″ 90°	$8\ (3\cdot8^2)$ $8\ (3\cdot4^3)$ $16\ (3\cdot4\cdot3\cdot8)$	[2, 4]

68	Augmented truncated dodecahedron	$Q_5\,T_5$	*5*·5 {3} 5 {4} 1 {5} 1+*2*·5 {10}	5 ⟨3·4⟩ 10 ⟨3·4⟩ 5 ⟨4·5⟩ 5 ⟨3·10⟩ *3*·5+*4*·10 ⟨3·10⟩ *3*·5+10 ⟨10·10⟩	174° 20′ 24″ 159° 5′ 41″ 148° 16′ 57″ 153° 56′ 33″ 142° 37′ 21″ 116° 33′ 54″	*4*·5+*3*·10 (3·10^2) 5 (3·4·5·4) 10 (3·4·3·10)	[5]
69	Parabiaugmented truncated dodecahedron	p-$Q_5^2T_5$	*3*·10 {3} 10 {4} 2 {5} 10 {10}	10 ⟨3·4⟩ 20 ⟨3·4⟩ 10 ⟨4·5⟩ 10 ⟨3·10⟩ 10+*2*·20 ⟨3·10⟩ *2*·10 ⟨10·10⟩	174° 20′ 24″ 159° 5′ 41″ 148° 16′ 57″ 153° 56′ 33″ 142° 37′ 21″ 116° 33′ 54″	*2*·10+20 (3·10^2) 10 (3·4·5·4) 20 (3·4·3·10)	[2$^+$, 10]
70	Metabiaugmented truncated dodecahedron	m-$Q_5^2T_5$	*5*·2+*5*·4 {3} 2+*2*·4 {4} 2 {5} *3*·2+4 {10}	2+*2*·4 ⟨3·4⟩ *5*·4 ⟨3·4⟩ 2+*2*·4 ⟨4·5⟩ 2+*2*·4 ⟨3·10⟩ *3*·2+*11*·4 ⟨3·10⟩ *2*+2+*4*·4 ⟨10·10⟩	174° 20′ 24″ 159° 5′ 41″ 148° 16′ 57″ 153° 56′ 33″ 142° 37′ 21″ 116° 33′ 54″	*4*·2+*8*·4 (3·10^2) 2+*2*·4 (3·4·5·4) *5*·4 (3·4·3·10)	[2]
71	Triaugmented truncated dodecahedron	$Q_5^3T_5$	*2*+*3*·3+*4*·6 {3} 3+*2*·6 {4} 3 {5} *3*·3 {10}	3+*2*·6 ⟨3·4⟩ *5*·6 ⟨3·4⟩ 3+*2*·6 ⟨4·5⟩ 3+*2*·6 ⟨3·10⟩ *3*·3+*6*·6 ⟨3·10⟩ *3*·3+6 ⟨10·10⟩	174° 20′ 24″ 159° 5′ 41″ 148° 16′ 57″ 153° 56′ 33″ 142° 37′ 21″ 116° 33′ 54″	*4*·3+*3*·6 (3·10^2) 3+*2*·6 (3·4·5·4) *5*·6 (3·4·3·10)	[3]
72	Gyrate rhombicosidodecahedron	$g\,E_5$	*4*·5 {3} *4*·5+10 {4} *2*+*2*·5 {5}	*3*·5+*4*·10 ⟨3·4⟩ 5 ⟨4·4⟩ 5 ⟨3·5⟩ *3*·5+*4*·10 ⟨4·5⟩	159° 5′ 41″ 153° 26′ 6″ 153° 56′ 33″ 148° 16′ 57″	10 (3·4^2·5) *4*·5+*3*·10 (3·4·5·4)	[5]
73	Parabigyrate rhombicosidodecahedron	p-g^2E_5	*2*·10 {3} *3*·10 {4} 2+10 {5}	10+*2*·20 ⟨3·4⟩ 10 ⟨4·4⟩ 10 ⟨3·5⟩ 10+*2*·20 ⟨4·5⟩	159° 5′ 41″ 153° 26′ 6″ 153° 56′ 33″ 148° 16′ 57″	20 (3·4^2·5) *2*·10+20 (3·4·5·4)	[2$^+$, 10]

TABLE III—*continued*

No.	Name	Symbol	Faces	Edges and dihedral angles		Vertices	Group
74	Metabigyrate rhombicosidodecahedron	*m*-g^2E_5	*4*·2+*3*·4 {3} *2*+*2*·2+*6*·4 {4} *4*·2+4 {5}	*3*·2+*11*·4 ⟨3·4⟩ 2+*2*·4 ⟨4·4⟩ 2+*2*·4 ⟨3·5⟩ *3*·2+*11*·4 ⟨4·5⟩	159° 5′ 41″ 153° 26′ 6″ 153° 56′ 33″ 148° 16′ 57″	5·4 (3·4²·5) 4·2+8·4 (3·4·5·4)	[2]
75	Trigyrate rhombicosidodecahedron	g^3E_5	*2*+*2*·3+*2*·6 {3} *4*·3+*3*·6 {4} *4*·3 {5}	*3*·3+*6*·6 ⟨3·4⟩ 3+*2*·6 ⟨4·4⟩ 3+*2*·6 ⟨3·5⟩ *3*·3+*6*·6 ⟨4·5⟩	159° 5′ 41″ 153° 26′ 6″ 153° 56′ 33″ 148° 16′ 57″	*5*·6 (3·4²·5) *4*·3+*3*·6 (3·4·5·4)	[3]
76	Diminished rhombicosidodecahedron	$Q_5^{-1}E_5$	*3*·5 {3} *3*·5+10 {4} 1+*2*·5 {5} 1 {10}	*3*·5+*3*·10 ⟨3·4⟩ *2*·5+*4*·10 ⟨4·5⟩ 5 ⟨4·10⟩ 5 ⟨5·10⟩	159° 5′ 41″ 148° 16′ 57″ 121° 43′ 3″ 116° 33′ 54″	10 (4·5·10) *3*·5+*3*·10 (3·4·5·4)	[5]
77	Paragyrate diminished rhombicosidodecahedron	*p*-g $Q_5^{-1}E_5$	*3*·5 {3} *3*·5+10 {4} 1+*2*·5 {5} 1 {10}	*2*·5+*3*·10 ⟨3·4⟩ 5 ⟨4·4⟩ 5 ⟨3·5⟩ 5+*4*·10 ⟨4·5⟩ 5 ⟨4·10⟩ 5 ⟨5·10⟩	159° 5′ 41″ 153° 26′ 6″ 153° 56′ 33″ 148° 16′ 57″ 121° 43′ 3″ 116° 33′ 54″	10 (4·5·10) 10 (3·4²·5) *3*·5+*2*·10 (3·4·5·4)	[5]
78	Metagyrate diminished rhombicosidodecahedron	*m*-g $Q_5^{-1}E_5$	*3*+*6*·2 {3} *3*+*11*·2 {4} *3*+*4*·2 {5} 1 {10}	*2*+*19*·2 ⟨3·4⟩ 1+*2*·2 ⟨4·4⟩ 1+*2*·2 ⟨3·5⟩ 1+*22*·2 ⟨4·5⟩ 1+*2*·2 ⟨4·10⟩ 1+*2*·2 ⟨5·10⟩	159° 5′ 41″ 153° 26′ 6″ 153° 56′ 33″ 148° 16′ 57″ 121° 43′ 3″ 116° 33′ 54″	*5*·2 (4·5·10) *5*·2 (3·4²·5) *3*+*16*·2 (3·4·5·4)	[ ]

79	Bigyrate diminished rhombicosidodecahedron	$g^2Q_5^{-1}E_5$	3+6·2 {3} 3+11·2 {4} 3+4·2 {5} 1 {10}	3+16·2 ⟨3·4⟩ 159° 5′ 41″ 5·2 ⟨4·4⟩ 153° 26′ 6″ 5·2 ⟨3·5⟩ 153° 56′ 33″ 2+19·2 ⟨4·5⟩ 148° 16′ 57″ 1+2·2 ⟨4·10⟩ 121° 43′ 3″ 1+2·2 ⟨5·10⟩ 116° 33′ 54″	5·2 (4·5·10) 10·2 $(3·4^2·5)$ 3+11·2 (3·4·5·4)	[ ]
80	Parabidiminished rhombicosidodecahedron	p-$Q_5^{-2}E_5$	10 {3} 2·10 {4} 10 {5} 2 {10}	10+20 ⟨3·4⟩ 159° 5′ 41″ 2·20 ⟨4·5⟩ 148° 16′ 57″ 10 ⟨4·10⟩ 121° 43′ 3″ 10 ⟨5·10⟩ 116° 33′ 54″	20 (4·5·10) 10+20 (3·4·5·4)	$[2^+, 10]$
81	Metabidiminished rhombicosidodecahedron	m-$Q_5^{-2}E_5$	3·2+4 {3} 2+2+4·4 {4} 3·2+4 {5} 2 {10}	3·2+6·4 ⟨3·4⟩ 159° 5′ 41″ 2·2+9·4 ⟨4·5⟩ 148° 16′ 57″ 2+2·4 ⟨4·10⟩ 121° 43′ 3″ 2+2·4 ⟨5·10⟩ 116° 33′ 54″	5·4 (4·5·10) 3·2+6·4 (3·4·5·4)	[2]
82	Gyrate bidiminished rhombicosidodecahedron	$g\ Q_5^{-2}E_5$	4+3·2 {3} 4+8·2 {4} 4+3·2 {5} 2 {10}	3+11·2 ⟨3·4⟩ 159° 5′ 41″ 1+2·2 ⟨4·4⟩ 153° 26′ 6″ 1+2·2 ⟨3·5⟩ 153° 56′ 33″ 3+16·2 ⟨4·5⟩ 148° 16′ 57″ 5·2 ⟨4·10⟩ 121° 43′ 3″ 5·2 ⟨5·10⟩ 116° 33′ 54″	10·2 (4·5·10) 5·2 $(3·4^2·5)$ 4+8·2 (3·4·5·4)	[ ]
83	Tridiminished rhombicosidodecahedron	$Q_5^{-3}E_5$	2+3 {3} 3·3+6 {4} 3·3 {5} 3 {10}	3·3+6 ⟨3·4⟩ 159° 5′ 41″ 2·3+4·6 ⟨4·5⟩ 148° 16′ 57″ 3+2·6 ⟨4·10⟩ 121° 43′ 3″ 3+2·6 ⟨5·10⟩ 116° 33′ 54″	5·6 (4·5·10) 3·3+6 (3·4·5·4)	[3]
84	Snub disphenoid	$s\ S_2$	4+8 {3}	4 ⟨3·3⟩ 166° 26′ 26″ 8 ⟨3·3⟩ 121° 44′ 35″ 2+4 ⟨3·3⟩ 96° 11′ 54″	4 (3^4) 4 (3^5)	$[2^+, 4]$
85	Snub square antiprism	$s\ S_4$	8+16 {3} 2 {4}	8 ⟨3·3⟩ 164° 15′ 27″ 16 ⟨3·3⟩ 144° 8′ 37″ 8 ⟨3·3⟩ 114° 38′ 43″ 8 ⟨3·4⟩ 145° 26′ 26″	8 (3^5) 8 $(3^4·4)$	$[2^+, 8]$

TABLE III—*concluded*

No.	Name	Symbol	Faces	Edges and dihedral angles		Vertices	Group
86	Sphenocorona	$V_2 N_2$	2·2+2·4 {3} 2 {4}	2 ⟨3·3⟩ 4 ⟨3·3⟩ 4 ⟨3·3⟩ 1 ⟨3·3⟩ 4 ⟨3·3⟩ 4 ⟨3·4⟩ 2 ⟨3·4⟩ 1 ⟨4·4⟩	159° 53′ 33″ 143° 28′ 43″ 135° 59′ 30″ 131° 26′ 30″ 118° 53′ 32″ 109° 31′ 27″ 97° 27′ 20″ 117° 1′ 8″	4 ($3^3 \cdot 4$) 2 ($3^2 \cdot 4^2$) 2 (3^5) 2 (3^5)	[2]
87	Augmented sphenocorona	$Y_4 V_2 N_2$	4+6·2 {3} 1 {4}	2 ⟨3·3⟩ 2 ⟨3·3⟩ 1 ⟨3·3⟩ 2·2 ⟨3·3⟩ 2·2 ⟨3·3⟩ 1 ⟨3·3⟩ 2·2 ⟨3·3⟩ 2·2 ⟨3·3⟩ 1 ⟨3·4⟩ 2 ⟨3·4⟩ 1 ⟨3·4⟩	164° 15′ 35″ 159° 53′ 33″ 152° 11′ 28″ 143° 28′ 43″ 135° 59′ 30″ 131° 26′ 30″ 118° 53′ 32″ 109° 28′ 16″ 171° 45′ 17″ 109° 31′ 27″ 97° 27′ 20″	1 (3^4) 2 ($3^3 \cdot 4$) 2 (3^5) 2 (3^5) 2 (3^5) 2 ($3^4 \cdot 4$)	[ ]
88	Sphenomegacorona	$V_2 M_2$	2·2+3·4 {3} 2 {4}	4 ⟨3·3⟩ 1 ⟨3·3⟩ 4 ⟨3·3⟩ 4 ⟨3·3⟩ 4 ⟨3·3⟩ 2·2 ⟨3·3⟩ 4 ⟨3·4⟩ 2 ⟨3·4⟩ 1 ⟨4·4⟩	171° 38′ 45″ 161° 28′ 58″ 143° 44′ 18″ 129° 26′ 40″ 117° 21′ 20″ 86° 43′ 37″ 154° 43′ 20″ 137° 14′ 24″ 72° 58′ 23″	2 (3^4) 2 ($3^2 \cdot 4^2$) 2 (3^5) 2 (3^5) 4 ($3^4 \cdot 4$)	[2]

89	Hebesphenomegacorona	$U_2 M_2$	*3*·2+*3*·4 {3} 1+2 {4}	*2*·4 ⟨3·3⟩ 1 ⟨3·3⟩ 4 ⟨3·3⟩ *2*·4 ⟨3·3⟩ 2 ⟨3·3⟩ 2 ⟨3·4⟩ 2+4 ⟨3·4⟩ 2 ⟨4·4⟩	157° 8′ 53″ 149° 33′ 53″ 141° 20′ 28″ 128° 29′ 46″ 111° 44′ 5″ 152° 58′ 32″ 133° 58′ 22″ 102° 31′ 25″	4 ($3^2·4^2$) 2 (3^5) *2*·2 (3^5) 4 ($3^4·4$)	[2]
90	Disphenocingulum	$V_2^2G_2$	4+*2*·8 {3} 4 {4}	4 ⟨3·3⟩ 8 ⟨3·3⟩ 8 ⟨3·3⟩ 4 ⟨3·3⟩ 4 ⟨3·4⟩ 8 ⟨3·4⟩ 2 ⟨4·4⟩	166° 48′ 41″ 148° 26′ 2″ 133° 35′ 28″ 124° 42′ 7″ 154° 25′ 8″ 136° 20′ 9″ 100° 11′ 38″	4 ($3^2·4^2$) 4 (3^5) 8 ($3^4·4$)	[2^+, 4]
91	Bilunabirotunda	$L_2^2R_2^2$	*2*·4 {3} 2 {4} 4 {5}	4 ⟨3·4⟩ 4 ⟨3·4⟩ 8 ⟨3·5⟩ 8 ⟨3·5⟩ 2 ⟨5·5⟩	159° 5′ 41″ 110° 54′ 19″ 142° 37′ 21″ 100° 48′ 44″ 63° 26′ 6″	4 ($3·5^2$) 8 (3·4·3·5) 2 (3·5·3·5)	[2, 2]
92	Triangular hebesphenorotunda	$U_3 R_3$	1+*2*·3+6 {3} 3 {4} 3 {5} 1 {6}	6 ⟨3·3⟩ 6 ⟨3·4⟩ 3 ⟨3·4⟩ 3+6 ⟨3·5⟩ 6 ⟨3·5⟩ 3 ⟨3·6⟩ 3 ⟨4·6⟩	138° 11′ 23″ 159° 5′ 41″ 110° 54′ 19″ 142° 37′ 21″ 100° 48′ 44″ 138° 11′ 23″ 110° 54′ 19″	3 ($3^3·5$) 6 (3·4·3·5) 3 (3·5·3·5) 6 ($3^2·4·6$)	[3]

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